1 Computation of first and second derivatives for the paper "Asset Allocation by Variance Sensitivity Analysis"

1.1 Notation

$y_{t,i}$ $\quad i = 1, ..., N$ $\quad$ asset returns
$a = [a_1, ..., a_{N-1}]'$ $\quad$ portfolio weigths
$y_{t\backslash N} = [y_{t,1}, ..., y_{t,N-1}]'$
$Y_t = a'y_{t\backslash N} + (1 - a'\iota)y_{t,N}$ $\quad$ portfolio returns

GARCH model:
$Y_t = \sqrt{h_t}\varepsilon_t$ $\quad \varepsilon_t \tilde{} i.i.d.N(0,1)$
$h_t = z_t'\theta$ $\quad z_t = [1, Y_{t-1}^2, h_{t-1}]'$, $\quad \theta = [\omega, \alpha, \beta]'$

In what follows we follow the conventions and rules on matrix differentiation as in Appendix A.13 of Helmut Lutkepohl (1990), Introduction to Multiple Time Series Analysis, Springer-Verlag.

1.2 Compute the first derivative (equation 3 in text)

In the rest of these pages, we indicate the explicit dependence of θ on a by $\theta(a)$. Whenever this dependence is not made explicit, it means that we treat θ as not depending on a.

$$\frac{\partial h_t(\theta(a))}{\partial a} = \frac{\partial z_t(\theta(a))'}{\partial a}\theta + \frac{\partial \theta(a)'}{\partial a} z_t \tag{1}$$

Compute $\frac{\partial z_t(\theta(a))}{\partial a}$ first.

$$\frac{\partial z_t(\theta(a))'}{\partial a} = \begin{bmatrix} 0_{(N-1\times 1)} & \frac{\partial Y_{t-1}^2}{\partial a} & \frac{\partial h_{t-1}(\theta(a))}{\partial a} \end{bmatrix} \tag{2}$$

where

$$\frac{\partial Y_{t-1}^2}{\partial a} = 2Y_{t-1}(y_{t-1\backslash N} - \iota y_{t-1,N}) \tag{3}$$

and $\frac{\partial h_{t-1}(\theta(a))}{\partial a}$ can be computed recursively.

To compute $\frac{\partial \theta(a)'}{\partial a}$ we apply Theorem 1:

$$\frac{\partial \theta(a)'}{\partial a} = -\underbrace{(I_{\theta a})'}_{(N-1\times 3)}\underbrace{(I_{\theta\theta})^{-1}}_{(3\times 3)} \tag{4}$$

where

$$I_{\theta\theta} = \sum_{t=1}^{T} \frac{\partial^2 l_t}{\partial\theta\partial\theta'} \tag{5}$$

$$I_{\theta a} = \sum_{t=1}^{T} \frac{\partial^2 l_t}{\partial\theta\partial a'} \tag{6}$$

Note that $l_t = -0.5[\ln(h_t) + Y_t^2 h_t{}^{-1}]$ and

$$\frac{\partial l_t}{\partial\theta} = -0.5\frac{\partial h_t}{\partial\theta}H \tag{7}$$

where

$$H \equiv \left(h_t^{-1} - Y_t^2 h_t^{-2}\right) \tag{8}$$

$$\frac{\partial h_t}{\partial\theta} = z_t + \frac{\partial z_t'}{\partial\theta}\theta \tag{9}$$

$$\frac{\partial z_t'}{\partial\theta} = \begin{bmatrix} 0_{(3\times 2)} & \frac{\partial h_{t-1}}{\partial\theta} \end{bmatrix} \tag{10}$$

Therefore $\frac{\partial^2 l_t}{\partial\theta\partial\theta'}$ can be computed as follows:

$$\frac{\partial^2 l_t}{\partial\theta\partial\theta'} = -0.5\left[\frac{\partial h_t}{\partial\theta}\frac{\partial H}{\partial\theta'} + H\frac{\partial^2 h_t}{\partial\theta\partial\theta'}\right] \tag{11}$$

where

$$\frac{\partial H}{\partial\theta'} = \frac{\partial h_t}{\partial\theta'}\tilde{H} \tag{12}$$

$$\tilde{H} \equiv \left(-h_t^{-2} + 2Y_t^2 h_t^{-3}\right) \tag{13}$$

$$\frac{\partial^2 h_t}{\partial\theta\partial\theta'} = \frac{\partial z_t}{\partial\theta'} + \frac{\partial z_t'}{\partial\theta} + \left(\theta' \otimes I_3\right)\frac{\partial}{\partial\theta'}vec\left(\frac{\partial z_t'}{\partial\theta}\right) \tag{14}$$

$$\frac{\partial}{\partial\theta'}vec\left(\frac{\partial z_t'}{\partial\theta}\right) = \begin{bmatrix} 0_{(6\times 3)} \\ \frac{\partial^2 h_{t-1}}{\partial\theta\partial\theta'} \end{bmatrix} \equiv G \tag{15}$$

Analogously, $\frac{\partial^2 l_t}{\partial\theta\partial a'}$ is given by:

$$\frac{\partial^2 l_t}{\partial\theta\partial a'} = -0.5\left[\frac{\partial h_t}{\partial\theta}\frac{\partial H}{\partial a'} + H\frac{\partial^2 h_t}{\partial\theta\partial a'}\right] \tag{16}$$

where

$$\frac{\partial H}{\partial a'} = \frac{\partial h_t}{\partial a'}\tilde{H} - h_t^{-2}\frac{\partial Y_t^2}{\partial a'} \tag{17}$$

$$\frac{\partial h_t}{\partial a'} = \theta'\frac{\partial z_t}{\partial a'} \tag{18}$$

$$\frac{\partial z_t}{\partial a'} = \begin{bmatrix} 0_{(1\times N-1)} \\ \frac{\partial Y^2_{t-1}}{\partial a'} \\ \frac{\partial h_{t-1}}{\partial a'} \end{bmatrix} \tag{19}$$

$$\frac{\partial^2 h_t}{\partial\theta\partial a'} = \frac{\partial z_t}{\partial a'} + \left(\theta' \otimes I_3\right) \frac{\partial}{\partial a'} vec\left(\frac{\partial z'_t}{\partial\theta}\right) \tag{20}$$

$$\frac{\partial}{\partial a'} vec\left(\frac{\partial z'_t}{\partial\theta}\right) = \begin{bmatrix} 0_{(6\times N-1)} \\ \frac{\partial^2 h_{t-1}}{\partial\theta\partial a'} \end{bmatrix} \equiv \underset{(9\times N-1)}{F} \tag{21}$$

1.3 Compute the second derivative (equation 4 in text)

$$\begin{aligned} \frac{\partial^2 h_t(\theta(a))}{\partial a\partial a'} = & \frac{\partial z_t(\theta(a))'}{\partial a}\frac{\partial\theta(a)}{\partial a'} + \frac{\partial\theta(a)'}{\partial a}\frac{\partial z_t(\theta(a))}{\partial a'} + \\ & + \left(\theta' \otimes I_{N-1}\right)\frac{\partial}{\partial a'} vec\left(\frac{\partial z_t(\theta(a))'}{\partial a}\right) + \\ & + \left(z_t\left(\theta\left(a\right)\right)' \otimes I_{N-1}\right)\frac{\partial}{\partial a'} vec\left(\frac{\partial\theta(a)'}{\partial a}\right) \end{aligned} \tag{22}$$

We need to compute only $\frac{\partial}{\partial a'} vec\left(\frac{\partial z_t(\theta(a))'}{\partial a}\right)$ and $\frac{\partial}{\partial a'} vec\left(\frac{\partial\theta(a)'}{\partial a}\right)$, since $\frac{\partial z_t(\theta(a))'}{\partial a}$ has already been computed in (2) and $\frac{\partial\theta(a)'}{\partial a}$ in (4):

$$\frac{\partial}{\partial a'} vec\left(\frac{\partial z_t\left(\theta\left(a\right)\right)'}{\partial a}\right) = \begin{bmatrix} 0_{(N-1\times N-1)} \\ \frac{\partial^2 Y^2_{t-1}}{\partial a\partial a'} \\ \frac{\partial^2 h_{t-1}(\theta(a))}{\partial a\partial a'} \end{bmatrix} \tag{23}$$

where

$$\frac{\partial^2 Y^2_{t-1}}{\partial a\partial a'} = 2(y_{t-1\backslash N} - \iota y_{t-1,N})(y_{t-1\backslash N} - \iota y_{t-1,N})' \tag{24}$$

and $\frac{\partial^2 h_{t-1}(\theta(a))}{\partial a\partial a'}$ can be computed recursively.

To compute $\frac{\partial}{\partial a'} vec\left(\frac{\partial\theta(a)'}{\partial a}\right)$ note first that if A is a $(p\times p)$ symmetric non-singular matrix $\frac{\partial vec\left(AA^{-1}\right)}{\partial a'} = (I_p \otimes A)\frac{\partial vec\left(A^{-1}\right)}{\partial a'} + \left(A^{-1} \otimes I_p\right)\frac{\partial vec(A)}{\partial a'} = 0_{(p^2\times N-1)}$. Therefore, since $(I_p \otimes A)^{-1} = \left(I_p \otimes A^{-1}\right)$, $\frac{\partial vec\left(A^{-1}\right)}{\partial a'} = -\left(I_p \otimes A^{-1}\right)\left(A^{-1} \otimes I_p\right)\frac{\partial vec(A)}{\partial a'} = -\left(A^{-1} \otimes A^{-1}\right)\frac{\partial vec(A)}{\partial a'}$. We also apply the following property of the vec operator: If A is an $(m\times n)$ matrix, then $vec(A') = K_{m,n} vec(A)$, where $K_{m,n}$ is the $(mn\times mn)$ commutation matrix.

$$\frac{\partial}{\partial a'} vec\left(\frac{\partial\theta(a)'}{\partial a}\right) = -\left\{\left(I_3 \otimes I'_{\theta a}\right)\frac{\partial}{\partial a'} vec\left[(I_{\theta\theta})^{-1}\right] + \left[(I_{\theta\theta})^{-1} \otimes I_{N-1}\right]\frac{\partial}{\partial a'} vec\left(I'_{\theta a}\right)\right\}$$

$$= -\left\{(I_3 \otimes I'_{\theta a})\left[-\left((I_{\theta\theta})^{-1} \otimes (I_{\theta\theta})^{-1}\right)\right] \tfrac{\partial}{\partial a'} vec\,[I_{\theta\theta}] + \left[(I_{\theta\theta})^{-1} \otimes I_{N-1}\right] \tfrac{\partial}{\partial a'} vec\,(I'_{\theta a})\right\}$$
$$= -\left\{-\left((I_{\theta\theta})^{-1} \otimes I'_{\theta a}(I_{\theta\theta})^{-1}\right) \tfrac{\partial}{\partial a'} vec\,[I_{\theta\theta}] + \left[(I_{\theta\theta})^{-1} \otimes I_{N-1}\right] \tfrac{\partial}{\partial a'} vec\,(I'_{\theta a})\right\}$$
$$= -\left\{-\left((I_{\theta\theta})^{-1} \otimes I'_{\theta a}(I_{\theta\theta})^{-1}\right) \tfrac{\partial}{\partial a'} vec\,[I_{\theta\theta}] + \left[(I_{\theta\theta})^{-1} \otimes I_{N-1}\right] \tfrac{\partial}{\partial a'} vec\,(I'_{\theta a})\right\}$$

$$\begin{aligned} \tfrac{\partial}{\partial a'} vec\left(\tfrac{\partial \theta(a)'}{\partial a}\right) = & -\left((I_{\theta\theta})^{-1} \otimes \tfrac{\partial \theta(a)'}{\partial a}\right) \tfrac{\partial}{\partial a'} vec\,[I_{\theta\theta}\,(\theta\,(a))] - \\ & - \left[(I_{\theta\theta})^{-1} \otimes I_{N-1}\right] \tfrac{\partial}{\partial a'} vec\,(I'_{\theta a}\,(\theta\,(a))) \end{aligned} \tag{25}$$

We need to compute $\frac{\partial}{\partial a'} vec\,[I_{\theta\theta}\,(\theta\,(a))]$ and $\frac{\partial}{\partial a'} vec\,(I'_{\theta a}\,(\theta\,(a)))$. Compute $\frac{\partial}{\partial a'} vec\left(\frac{\partial^2 l_t}{\partial\theta\partial\theta'}\,(\theta\,(a))\right)$ first:

$$\begin{aligned} \tfrac{\partial}{\partial a'} vec\left(\tfrac{\partial^2 l_t}{\partial\theta\partial\theta'}\,(\theta\,(a))\right) = -0.5\,\Big\{ & \left(I_3 \otimes \tfrac{\partial h_t}{\partial\theta}\right) \tfrac{\partial}{\partial a'}\left(\tfrac{\partial H}{\partial\theta}\,(\theta\,(a))\right) + \\ & + \left(\tfrac{\partial H}{\partial\theta} \otimes I_3\right) \tfrac{\partial}{\partial a'}\left(\tfrac{\partial h_t}{\partial\theta}\,(\theta\,(a))\right) + \\ & + vec\left(\tfrac{\partial^2 h_t}{\partial\theta\partial\theta'}\right) \tfrac{\partial H(\theta(a))}{\partial a'} + \\ & + H \tfrac{\partial}{\partial a'}\left[vec\left(\tfrac{\partial^2 h_t}{\partial\theta\partial\theta'}\,(\theta\,(a))\right)\right]\Big\} \end{aligned} \tag{26}$$

where $\frac{\partial h_t}{\partial\theta}$ has been computed in (9) and $\frac{\partial H}{\partial\theta'}$ in (12).

To evaluate this function we need to compute $\frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial\theta}\,(\theta\,(a))\right)$, $\frac{\partial H(\theta(a))}{\partial a'}$, $\frac{\partial}{\partial a'}\left(\frac{\partial H}{\partial\theta}\,(\theta\,(a))\right)$ and $\frac{\partial}{\partial a'}\left[vec\left(\frac{\partial^2 h_t}{\partial\theta\partial\theta'}\,(\theta\,(a))\right)\right]$. For $\frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial\theta}\,(\theta\,(a))\right)$, we have:

$$\frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial\theta}\,(\theta\,(a))\right) = \frac{\partial z_t\,(\theta\,(a))}{\partial a'} + \frac{\partial z'_t}{\partial\theta}\frac{\partial\theta(a)}{\partial a'} + \left(\theta' \otimes I_3\right)\frac{\partial}{\partial a'} vec\left(\frac{\partial z'_t}{\partial\theta}\,(\theta\,(a))\right) \tag{27}$$

where $\frac{\partial z_t(\theta(a))'}{\partial a}$ has been computed in (2), $\frac{\partial z'_t}{\partial\theta}$ in (10), $\frac{\partial h_t}{\partial\theta}$ in (9),

$$\frac{\partial}{\partial a'} vec\left(\frac{\partial z'_t}{\partial\theta}\,(\theta\,(a))\right) = \begin{bmatrix} 0_{(6\times N-1)} \\ \frac{\partial}{\partial a'}\left(\frac{\partial h_{t-1}}{\partial\theta}\,(\theta\,(a))\right) \end{bmatrix} \tag{28}$$

and the last term can be computed recursively.

For $\frac{\partial H(\theta(a))}{\partial a'}$, we have:

$$\frac{\partial H\,(\theta\,(a))}{\partial a'} = \frac{\partial h_t\,(\theta\,(a))}{\partial a'}\tilde{H} - h_t^{-2}\frac{\partial Y_t^2}{\partial a'} \tag{29}$$

where $\frac{\partial h_t(\theta(a))}{\partial a'}$ has been computed in (1).

For $\frac{\partial}{\partial a'}\left(\frac{\partial H}{\partial\theta}\,(\theta\,(a))\right)$, we have:

$$\frac{\partial}{\partial a'}\left(\frac{\partial H}{\partial\theta}\,(\theta\,(a))\right) = \frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial\theta}\,(\theta\,(a))\right)\tilde{H} + \frac{\partial h_t}{\partial\theta}\frac{\partial\tilde{H}\,(\theta\,(a))}{\partial a'} \tag{30}$$

where

$$\frac{\partial\tilde{H}\,(\theta\,(a))}{\partial a'} = \left(2h_t^{-3} - 6Y_t^2 h_t^{-4}\right)\frac{\partial h_t\,(\theta\,(a))}{\partial a'} + 2h_t^{-3}\frac{\partial Y_t^2}{\partial a'} \tag{31}$$

For $\frac{\partial}{\partial a'}\left[vec\left(\frac{\partial^2 h_t}{\partial\theta\partial\theta'}\left(\theta\left(a\right)\right)\right)\right]$, note first that $\frac{\partial^2 h_t}{\partial\theta\partial\theta'} = \frac{\partial^2 h_t}{\partial\theta'\partial\theta} = \frac{\partial z_t'}{\partial\theta} + \frac{\partial z_t}{\partial\theta'} + \frac{\partial}{\partial\theta}vec\left(\frac{\partial z_t}{\partial\theta'}\right)'\left(I_3\otimes\theta\right)$. Therefore:

$$\begin{aligned}\frac{\partial}{\partial a'}\left[vec\left(\frac{\partial^2 h_t}{\partial\theta'\partial\theta}\left(\theta\left(a\right)\right)\right)\right] = {} & \frac{\partial}{\partial a'}vec\left(\frac{\partial z_t'}{\partial\theta}\left(\theta\left(a\right)\right)\right) + \\ & + \frac{\partial}{\partial a'}vec\left(\frac{\partial z_t}{\partial\theta'}\left(\theta\left(a\right)\right)\right) + \\ & + \frac{\partial}{\partial a'}vec\left[\frac{\partial}{\partial\theta}vec\left(\frac{\partial z_t}{\partial\theta'}\right)'\left(I_3\otimes\theta\right)\right]\end{aligned} \tag{32}$$

For $\frac{\partial}{\partial a'}vec\left(\frac{\partial z_t}{\partial\theta'}\left(\theta\left(a\right)\right)\right)$, note that since $vec\left(\frac{\partial z_t}{\partial\theta'}\left(\theta\left(a\right)\right)\right) = K_{3,3}vec\left(\frac{\partial z_t'}{\partial\theta}\left(\theta\left(a\right)\right)\right)$, we have:

$$\frac{\partial}{\partial a'}vec\left(\frac{\partial z_t}{\partial\theta'}\left(\theta\left(a\right)\right)\right) = K_{3,3}\left[\frac{\partial}{\partial a'}vec\left(\frac{\partial z_t'}{\partial\theta}\left(\theta\left(a\right)\right)\right)\right] \tag{33}$$

It remains to compute $\frac{\partial}{\partial a'}vec\left[\frac{\partial}{\partial\theta}vec\left(\frac{\partial z_t}{\partial\theta'}\right)'\left(I_3\otimes\theta\right)\right]$:

$$\begin{aligned}\frac{\partial}{\partial a'}vec\left[\frac{\partial}{\partial\theta}vec\left(\frac{\partial z_t}{\partial\theta'}\right)'\left(I_3\otimes\theta\right)\right] = {} & \left(I_3\otimes G'\right)\frac{\partial}{\partial a'}vec\left(I_3\otimes\theta\right) + \\ & + \left[\left(I_3\otimes\theta'\right)\otimes I_3\right]\frac{\partial}{\partial a'}vec\left(G'\right)\end{aligned} \tag{34}$$

where, using property (27) of vec in Lutkepohl:

$$L_1 \equiv \frac{\partial}{\partial a'}vec\left(I_3\otimes\theta\right) = \left(I_3\otimes K_{1,3}\otimes I_3\right)\left(vec(I_3)\otimes\frac{\partial\theta(a)}{\partial a'}\right) \tag{35}$$

$$L_2 \equiv \frac{\partial}{\partial a'}vec(G') = \left[\begin{array}{c} 0_{(18\times N-1)} \\ \frac{\partial}{\partial a'}\left[vec\left(\frac{\partial^2 h_{t-1}}{\partial\theta\partial\theta'}\left(\theta\left(a\right)\right)\right)\right]\end{array}\right] \tag{36}$$

We now compute $\frac{\partial}{\partial a'}vec\left(\frac{\partial^2 l_t}{\partial\theta'\partial a}\left(\theta\left(a\right)\right)\right)$, the other missing term in (25):

$$\begin{aligned}\frac{\partial}{\partial a'}vec\left(\frac{\partial^2 l_t}{\partial\theta'\partial a}\left(\theta\left(a\right)\right)\right) = -0.5\Big\{ & \left(I_3\otimes\frac{\partial H}{\partial a}\right)\frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial\theta}\left(\theta\left(a\right)\right)\right) + \\ & + \left(\frac{\partial h_t}{\partial\theta}\otimes I_{N-1}\right)\frac{\partial}{\partial a'}\left(\frac{\partial H}{\partial a}\left(\theta\left(a\right)\right)\right) + \\ & + vec\left(\frac{\partial^2 h_t}{\partial\theta'\partial a}\right)\frac{\partial H(\theta(a))}{\partial a'} + \\ & + H\frac{\partial}{\partial a'}vec\left(\frac{\partial^2 h_t}{\partial\theta'\partial a}\left(\theta\left(a\right)\right)\right)\Big\}\end{aligned} \tag{37}$$

where $\frac{\partial H}{\partial a'}$ has been computed in (17), $\frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial\theta}\left(\theta\left(a\right)\right)\right)$ in (27), $\frac{\partial h_t}{\partial\theta}$ in (9), $\frac{\partial H(\theta(a))}{\partial a'}$ in (29) and $\frac{\partial^2 h_t}{\partial\theta'\partial a}$ in (20).

We need to compute $\frac{\partial}{\partial a'}\left(\frac{\partial H}{\partial a}\left(\theta\left(a\right)\right)\right)$ and $\frac{\partial}{\partial a'}vec\left(\frac{\partial^2 h_t}{\partial\theta'\partial a}\left(\theta\left(a\right)\right)\right)$.

$$\frac{\partial}{\partial a'}\left(\frac{\partial H}{\partial a}\left(\theta\left(a\right)\right)\right) = \tilde{H}\frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial a}\left(\theta\left(a\right)\right)\right) + \frac{\partial h_t}{\partial a}\frac{\partial\tilde{H}\left(\theta\left(a\right)\right)}{\partial a'} - h_t^{-2}\frac{\partial^2 Y_t^2}{\partial a\partial a'} + 2h_t^{-3}\frac{\partial Y_t^2}{\partial a}\frac{\partial h_t\left(\theta\left(a\right)\right)}{\partial a'} \tag{38}$$

where

$$\frac{\partial}{\partial a'}\left(\frac{\partial h_t}{\partial a}\left(\theta\left(a\right)\right)\right)=\frac{\partial z_t'}{\partial a}\frac{\partial \theta(a)}{\partial a'}+\left(\theta'\otimes I_{N-1}\right)\frac{\partial}{\partial a'}vec\left(\frac{\partial z_t'}{\partial a}\left(\theta\left(a\right)\right)\right) \tag{39}$$

$$\frac{\partial}{\partial a'}vec\left(\frac{\partial z_t'}{\partial a}\left(\theta\left(a\right)\right)\right)=\left[\begin{array}{c} 0_{(N-1)\times(N-1)} \\ \frac{\partial^2 Y_{t-1}^2}{\partial a \partial a'} \\ \frac{\partial}{\partial a'}\left(\frac{\partial h_{t-1}}{\partial a}\left(\theta\left(a\right)\right)\right) \end{array}\right] \tag{40}$$

Finally, $\frac{\partial}{\partial a'}vec\left(\frac{\partial^2 h_t}{\partial \theta' \partial a}\left(\theta\left(a\right)\right)\right)=\frac{\partial}{\partial a'}vec\left(\frac{\partial z_t'}{\partial a}+F\left(\theta\left(a\right)\right)'\left(\theta(a)\otimes I_3\right)\right)$:

$$\begin{aligned} \tfrac{\partial}{\partial a'}vec\left(\tfrac{\partial^2 h_t}{\partial \theta' \partial a}\left(\theta(a)\right)\right) = \tfrac{\partial}{\partial a'}vec\left(\tfrac{\partial z_t'}{\partial a}\left(\theta(a)\right)\right)+ \\ +\left(I_3\otimes F'\right)\tfrac{\partial}{\partial a'}vec\left(\theta(a)\otimes I_3\right)+ \\ +\left[\left(\theta'\otimes I_3\right)\otimes I_{N-1}\right]\tfrac{\partial}{\partial a'}vec\left(F\left(\theta\left(a\right)\right)'\right) \end{aligned} \tag{41}$$

where, using again property (27) of vec in Lutkepohl:

$$J_1 \equiv \frac{\partial}{\partial a'}vec\left(\theta(a)\otimes I_3\right)=\left(K_{3,3}\otimes I_3\right)\left(\frac{\partial \theta(a)}{\partial a'}\otimes vec\left(I_3\right)\right) \tag{42}$$

$$J_2 \equiv \frac{\partial}{\partial a'}vec\left(F\left(\theta\left(a\right)\right)'\right)=\left[\begin{array}{c} 0_{6(N-1)\times(N-1)} \\ \frac{\partial}{\partial a'}vec\left(\frac{\partial^2 h_{t-1}}{\partial \theta' \partial a}\left(\theta\left(a\right)\right)\right) \end{array}\right] \tag{43}$$

We now have all the elements to compute the first and second derivatives of the GARCH(1,1) variances with respect to the portfolio weigths.